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LINEAR STABILITY OF PLANE COUETTE FLOW OF AN UPPER
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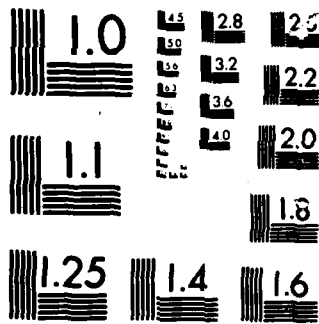
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LINEAR STABILITY OF PLANE COUETTE FLOW OF AN UPPER CONVECTED MAXWELL FLUID

Michael Renardy^{1,2} and Yuriko Renardy¹

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Abstract

This document
We investigate the linear stability of plane Couette flow of an upper convected Maxwell fluid using a spectral method to compute the eigenvalues. No instabilities are found. This is in agreement with the results of Ho and Denn [1] and Lee and Finlayson [2], but contradicts "proofs" of instability given by Gorodtsov and Leonov [3] and Akbay and Frischmann [4,5]. The errors in those arguments are pointed out. We also find that the numerical discretization can generate artificial instabilities (see also [1,6]). The qualitative behavior of the eigenvalue spectrum as well as the artificial instabilities is discussed.

AMS (MOS) Subject Classifications: 76A10, 76E05

Key Words: Upper convected Maxwell fluid, stability of parallel shear flow; *Shear stress*

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SIGNIFICANCE AND EXPLANATION

In the extrusion of polymers from a pipe, an instability known as melt fracture is frequently observed. There is considerable controversy concerning the question whether this instability develops in the inflow region, the outflow region or in the pipe itself and which physical effects are causing it. One of the hypotheses has been that it may arise from an instability in parallel shear flow, and that somehow a large ratio of the first normal stress difference to the shear stress (or a similar quantity) is responsible for causing instability. In this paper, the authors study a particular model of a viscoelastic fluid, and the results indicate that plane Couette flow of such a fluid is always linearly stable. The results therefore question the explanation of melt fracture mentioned above.

The paper also shows that artificial instabilities can arise from numerical discretization. This observation is relevant in the context of numerical simulations of flows of viscoelastic fluids, which have encountered many problems.

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LINEAR STABILITY OF PLANE COUETTE FLOW OF AN UPPER CONVECTED MAXWELL FLUID

Michael Renardy^{1,2} and Yuriko Renardy¹

1. Introduction

In polymer processing, instabilities known as melt fracture are frequently observed when the flow rate exceeds a critical value (see [7,8] for reviews). A satisfactory explanation for these instabilities has yet to be found, although there are a number of conjectures. There may be more than one mechanism leading to these instabilities. For example, there is some discussion whether the instability originates in parallel shear flow or in the inflow or outflow region. In either case, it is not understood what the physical mechanisms are and which features a fluid model must or must not have in order to predict the instability.

In this paper we study the linear stability of plane Couette flow of an upper convected Maxwell fluid. In our study we assume that stability can be determined from analyzing the eigenvalues of the differential equation. While this is well known to be true for the Navier-Stokes equations, there is no proof of this for the equations describing the upper convected Maxwell fluid. For a discussion of possible problems that can arise in a general context, see e.g. section 4.4 of [9].

Tlapa and Bernstein [10] have shown that Squire's theorem holds for the upper convected Maxwell fluid (it is well known [11] that this is not the case for non-Newtonian fluids in general), and therefore we can restrict our attention to two-dimensional disturbances. There is, however, a special class of disturbances for which Squire's transformation degenerates, and which therefore requires a separate investigation. An incorrect investigation of this class of disturbances, predicting instability, was given by Akbay and Frischmann [4,5]. We shall show in section 2 below that in fact these disturbances are always stable.

Gorodtsov and Leonov [3] give an exact analytical solution for the case of zero Reynolds number. They find that in addition to a stable continuous spectrum, there are exactly two eigenvalues for each value of α (the wave number in the streamwise direction). These eigenvalues are also stable. Since the perturbation introduced by a small Reynolds number is singular, it can not be concluded from this result that flows at small Reynolds number are stable. Gorodtsov and Leonov give an incorrect proof of instability at finite Reynolds numbers (see the discussion in section 2 below).

Denn and his coworkers [1,6,12] have studied the stability of plane Poiseuille flow of an upper convected Maxwell fluid using a numerical scheme based on the shooting method. They found [12] that viscoelastic effects are destabilizing at high Reynolds number and the

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critical Reynolds number is decreased by a small amount of elasticity (for Couette flow of a Newtonian fluid, there is no critical Reynolds number [13]). At low Reynolds numbers, no instabilities were found [1], but the numerical method led to artificial instabilities [1,6]. The recent study of Lee and Finlayson [2], which uses a similar numerical method to study both Poiseuille and Couette flow, confirms the absence of instabilities at low Reynolds numbers.

In this paper, we use a spectral method for the discretization and a matrix eigenvalue solver for computing the eigenvalues: i.e., in contrast to the earlier studies, we compute all the eigenvalues of the problem, rather than focussing on a few individual eigenvalues. This allows some insight into the overall qualitative behavior of the spectrum as well as the qualitative nature of artificial instabilities produced by the discretization. Again, no instabilities are found. In fact, the eigenvalues at low Reynolds number and high Weissenberg number appear to be rather uninteresting, although they are difficult to compute. The growth rates turn out to be close to the constant $-1/2W$ (W is the Weissenberg number). At high Reynolds number, we found that a small amount of elasticity has a destabilizing effect, but we did not find instability.

As far as the explanation of melt fracture is concerned, these results leave three possible conclusions:

- (a) Melt fracture does not originate in parallel shear flow.
- (b) Melt fracture is a finite amplitude effect which can not be explained by linear stability analysis.
- (c) Melt fracture can not be explained by the upper convected Maxwell model.

In connection with the last alternative, it may be relevant that many models, but not the upper convected Maxwell model, exhibit instabilities associated with a change of type [14]. Also, a corrected analysis for the special class of perturbations studied by Akbay and Frischmann seems to show instability (not associated with a change of type) for the lower convected Maxwell model (U. Akbay, private communication).

2. The governing equations and some analytical results

The flow considered is between two infinite parallel plates located at $y = \pm 1$, which are moving in the x -direction with velocities $+1$ and -1 . The equations, given in dimensionless form, are the equation of motion

$$R \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\nabla p + \text{div } \mathbf{T},$$

$$\text{div } \mathbf{u} = 0, \tag{1}$$

and the constitutive law for the upper convected Maxwell fluid

$$\mathbf{T} + W \left[\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{T} - (\nabla \mathbf{u}) \mathbf{T} - \mathbf{T} (\nabla \mathbf{u})^T \right] = \nabla \mathbf{u} + (\nabla \mathbf{u})^T. \tag{2}$$

Plane Couette flow is the trivial solution

$$\mathbf{u}_0 = (y, 0, 0), \quad \mathbf{T}_0 = \begin{pmatrix} 2W & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3)$$

To this we add a small perturbation

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}(y)e^{i\alpha x + i\beta z + \sigma t}, \quad \mathbf{T} = \mathbf{T}_0 + \mathbf{S}(y)e^{i\alpha x + i\beta z + \sigma t}, \quad (4)$$

and the equations are linearized with respect to the components of $\mathbf{v} = (u, v, w)$ and \mathbf{S} . Tlapa and Bernstein [10] show that Squire's transformation

$$(\alpha^*)^2 = \alpha^2 - \beta^2, \quad \beta^* = 0, \quad \sigma^*/\alpha^* = \sigma/\alpha, \quad \alpha^* u^* = \alpha u + \beta w, \quad v^* = v, \quad w^* = 0, \quad (5)$$

can be used to transform the problem to the two-dimensional case. However, this breaks down for the case where $\alpha u + \beta w = 0$, and hence this case requires a separate investigation. If $\alpha u + \beta w = 0$, it follows from the incompressibility condition

$$i\alpha u + v'(y) + i\beta w = 0, \quad (6)$$

and the boundary conditions at $y = \pm 1$ that $v = 0$, and equations (10) and (12) in [10] lead to

$$RS(\sigma + i\alpha y)w = w'' + 2i\alpha Ww' - (\alpha^2 + \beta^2 + 2\alpha^2 W^2)w, \quad (7)$$

where

$$S = 1 + W(\sigma + i\alpha y), \quad (8)$$

and an identical equation for u . In (7), we set $w = e^{-i\alpha W y} \phi$, and obtain

$$RS(\sigma + i\alpha y)\phi = \phi'' - (\alpha^2 + \beta^2 + \alpha^2 W^2)\phi. \quad (9)$$

For $\sigma = \beta = 0$, this agrees with equation (6a) in [5]. We now multiply (9) by $\bar{S}\bar{\phi}$, integrate from -1 to 1 and integrate by parts using the boundary conditions $\phi(1) = \phi(-1) = 0$. This yields

$$\begin{aligned} R \int_{-1}^1 (\sigma + i\alpha y) |S|^2 |\phi|^2 dy &= - \int_{-1}^1 \bar{S} |\phi'|^2 + \bar{S}' \phi' \bar{\phi} dy \\ &\quad - \int_{-1}^1 \bar{S} (\alpha^2 + \beta^2 + \alpha^2 W^2) |\phi|^2 dy. \end{aligned} \quad (10)$$

Taking real parts, we obtain

$$R \operatorname{Re} \sigma \int_{-1}^1 |S|^2 |\phi|^2 dy = -(1 + W \operatorname{Re} \sigma) \int_{-1}^1 |\phi'|^2 - (\alpha^2 + \beta^2 + \alpha^2 W^2) |\phi|^2 dy$$

$$-\alpha W \operatorname{Im} \int_{-1}^1 \phi' \bar{\phi} dy. \quad (11)$$

Using the inequality

$$\alpha W |\phi' \bar{\phi}| \leq \frac{1}{2} (|\phi'|^2 + \alpha^2 W^2 |\phi|^2), \quad (12)$$

we find that the right hand side of (11) is negative as long as $\operatorname{Re} \sigma \geq -1/(2W)$. Hence the left hand side must also be negative, and we conclude that $\operatorname{Re} \sigma$ must be negative.

We now turn to the study of two-dimensional disturbances. For this case, we introduce a stream function by $u = \phi'$, $v = -i\alpha\phi$. The equation for the stream function reads as follows (see [3,12]):

$$\begin{aligned} \phi^{(4)} + b_3(y)\phi''' + b_2(y)\phi'' + b_1(y)\phi' + b_0(y)\phi \\ = SR(yi\alpha + \sigma)(\phi'' - \alpha^2\phi), \end{aligned} \quad (13)$$

with boundary conditions

$$\phi(-1) = \phi'(-1) = \phi(1) = \phi'(1) = 0. \quad (14)$$

Here S is given by (8) and

$$\begin{aligned} b_3(y) &= 2i\alpha W \frac{S-1}{S}, \\ b_2(y) &= -2\alpha^2 - 2\alpha^2 W^2 \frac{(S-1)^2}{S^2}, \\ b_1(y) &= -2i\alpha^3 W \frac{S-1}{S} + 4i\alpha^3 W^3 \frac{S-1}{S^2}, \\ b_0(y) &= \alpha^4 + 2\alpha^4 W^2 \frac{S^2+1}{S^2} + \frac{4\alpha^4 W^4}{S^2}. \end{aligned} \quad (15)$$

An equivalent form is

$$\begin{aligned} [S^2 \frac{d^2}{dy^2} - 2i\alpha W S \frac{d}{dy} - 2\alpha^2 W^2 - \alpha^2 S^2] [\frac{d^2}{dy^2} + 2i\alpha W \frac{d}{dy} - \alpha^2 - 2\alpha^2 W^2] \phi \\ - S^3 R(i\alpha y + \sigma) (\frac{d^2}{dy^2} - \alpha^2) \phi = 0. \end{aligned} \quad (16)$$

The fact that the differential operator can be factored for $R = 0$ was used by Gorodtsov and Leonov [3] to obtain an exact solution for this case. The line segment from $-\frac{1}{W} - i\alpha$ to $-\frac{1}{W} + i\alpha$ is a continuous spectrum arising from the singular character of the equation when $S = 0$. Apart from that, there are exactly two eigenvalues for each value of α , for which Gorodtsov and Leonov give an exact expression.

Gorodtsov and Leonov also claim to find eigenvalues with positive real parts when $R \neq 0$. Their analysis is based on their equation (4.2) which they derive as an approximation:

$$\phi^{(4)} + 2i\alpha W \phi''' - 2\alpha^2(1 \pm \epsilon W^2 y \sqrt{2} - \epsilon W^2 \xi) \phi'' - 2\alpha^2 W (i\alpha \pm \epsilon W \sqrt{2}) \phi'$$

$$+\alpha^4(1 \pm 2\sqrt{2}\epsilon W^2 y - 2\epsilon W^2 \xi)\phi = 0. \quad (17)$$

Their instability proof is based on finding eigenvalues ξ with positive imaginary parts for this equation. In fact, however, all eigenvalues for (17) are real. To see this, we multiply (17) by $\bar{\phi}$ and integrate from -1 to 1 . After some integrations by parts, we obtain

$$\begin{aligned} \int_{-1}^1 \phi''^2 - 2i\alpha W \phi'' \bar{\phi}' + 2\alpha^2 |\phi'|^2 - 2i\alpha^3 W \phi' \bar{\phi} + \alpha^4 (1 \pm 2\sqrt{2}\epsilon W^2 y) |\phi|^2 dy \\ \pm 2\alpha^2 W^2 \epsilon \sqrt{2} \int_{-1}^1 y |\phi'|^2 dy - 2\alpha^2 W^2 \epsilon \xi \int_{-1}^1 \phi' \bar{\phi} + \alpha^2 |\phi|^2 dy = 0. \end{aligned} \quad (18)$$

Since all other terms in this equation are real, ξ must also be real.

For $\alpha = 0$, equation (13) reduces to

$$\phi^{(4)} = (1 + W\sigma)\sigma\phi''. \quad (19)$$

The case $\alpha = 0$ is a special case as far as boundary conditions are concerned, because $v = -i\alpha\phi = 0$ does not imply $\phi = 0$. However, the boundary condition $\phi = 0$ is obtained if the perturbation to the flow rate $\int_{-1}^1 u(y) dy$ is constrained to be zero and a uniform pressure gradient in the x -direction is allowed (alternatively, one can allow a nonzero flow rate, but no pressure gradient; in that case, one obtains a different boundary condition). Equation (19) with boundary conditions (14) has the eigenvalues

$$\sigma = \frac{-1 \pm \sqrt{1 + 4W\sigma^*/R}}{2W}, \quad (20)$$

where $\sigma^* = (1 + \sigma W)\sigma$ is given by $-n^2\pi^2$, $n \in N$ or by $-\beta^2$ where β is a nonzero root of $\beta = \tan\beta$. Except for a finite number, all roots given by (20) have real part $-\frac{1}{2W}$. Moreover, the convergence of the real parts to $-\frac{1}{2W}$ is most rapid when W/R is large.

3. Numerical Results

For the numerical solution, we multiplied equation (13) by S^2 , and discretized the equation and boundary conditions by the Chebyshev τ -method (see [15]). Since σ occurs in powers up to the fourth in the equation, this leads to a matrix eigenvalue problem of the form

$$\det(\sigma^4 A_4 + \sigma^3 A_3 + \sigma^2 A_2 + \sigma A_1 + A_0) = 0. \quad (21)$$

Since four equations contain the boundary conditions, the number of eigenvalues is $4(N-4)$ when N Chebyshev modes are used. We rewrite (21) as a first order equation with a matrix 4 times the size of the A_i , which is solved using the NAG routine F02GJF. This procedure allows us to compute all the eigenvalues rather than just individual ones. It is, however, quite sensitive to round-off error, and calculations at high values of αW had to be done in quadruple precision (on a VAX 11/780). In a few cases, we obtained improved accuracy

for individual eigenvalues by using a large number of Chebyshev modes and a Newton iteration based directly on (21). For this, quadruple precision is not necessary.

The program was tested against the exact result given by (20). We also compared our results with those of Lee and Finlayson [2]. They give a table listing eigenvalues computed for $R = 0.25$, $W = 1$, $\alpha = 15$ (in comparing their results with ours it must be noted that they consider Couette flow on the interval $[0,1]$ rather than $[-1,1]$ and that their eigenvalues are wave speeds rather than growth rates; we have transformed their results to our present notation). The eigenvalues of Lee and Finlayson were computed using an approximate equation. As Lee and Finlayson observe, this approximate equation produces the imaginary parts of the eigenvalues almost perfectly, but approximates the real parts rather poorly, leading to artificial instabilities at higher Weissenberg numbers. This is also reflected in the following comparison of our results (using 80 Chebyshev modes) with theirs.

Eigenvalues with lowest imaginary parts computed by our program	Eigenvalues given in table VI of [2]
$-0.85349 \pm 14.646i$	
$-0.50964 \pm 33.021i$	$-0.960 \pm 33.021i$
$-0.50824 \pm 36.766i$	
$-0.50694 \pm 39.783i$	
$-0.50579 \pm 42.414i$	
$-0.50480 \pm 44.793i$	$-0.834 \pm 44.796i$
$-0.50398 \pm 46.986i$	
$-0.50332 \pm 49.038i$	
$-0.50280 \pm 50.975i$	
$-0.50253 \pm 52.824i$	
$-0.50283 \pm 54.631i$	
$-0.50387 \pm 56.480i$	
$-0.50518 \pm 58.442i$	
$-0.50623 \pm 60.537i$	
$-0.50683 \pm 62.755i$	$-0.543 \pm 62.757i$
$-0.50700 \pm 65.074i$	
$-0.50685 \pm 67.479i$	
$-0.50646 \pm 69.956i$	
$-0.50595 \pm 72.494i$	
$-0.50536 \pm 75.085i$	$-0.432 \pm 75.084i$
$-0.50478 \pm 77.720i$	
$-0.50421 \pm 80.395i$	
$-0.50369 \pm 83.105i$	
$-0.50322 \pm 85.846i$	
$-0.50281 \pm 88.615i$	
$-0.50245 \pm 91.408i$	$-0.417 \pm 91.407i$
$-0.50214 \pm 94.224i$	$-0.420 \pm 94.224i$
$-0.50187 \pm 97.060i$	

$-0.50164 \pm 99.915i$	
$-0.50144 \pm 102.79i$	
$-0.50128 \pm 105.67i$	$-0.429 \pm 105.67i$
$-0.50119 \pm 108.58i$	
$-0.50135 \pm 111.49i$	
$-0.50236 \pm 114.42i$	
$-0.50579 \pm 117.37i$	
$-0.51427 \pm 120.35i$	$-0.441 \pm 120.30i$

The sparseness of the right column shows that the list of eigenvalues given by Lee and Finlayson is far from complete. As can be seen, the real parts of all our eigenvalues fall almost exactly on $-\frac{1}{2W}$, except for the first pair. This one pair of eigenvalues behaves in a qualitatively different fashion from the others. It corresponds to the two eigenvalues found by Gorodtsov and Leonov [3] for $R = 0$. The occurrence of the eigenvalues in complex conjugate pairs is due to the reflection symmetry of the Couette flow problem about the origin. For imaginary parts beyond 120, the numerical accuracy of our results deteriorates, as already evident in the deviation of the real parts of the last few eigenvalues in the table from $-\frac{1}{2W}$. We recomputed the last eigenvalues in the table above with 120 Chebyshev polynomials and found $-0.50074 \pm 120.30i$.

We did a number of calculations at $R = 1$, $\alpha = 1$ and various Weissenberg numbers. The following table shows the eigenvalues with the smallest imaginary parts.

$W = 0.2$	$W = 2$	$W = 20$	$W = 50$
$-4.9687 \pm 0.6462i$	$-0.36985 \pm 0.7740i$	$-0.026286 \pm 0.9750i$	$-0.010207 \pm 0.9900i$
$-2.5311 \pm 6.3370i$	$-0.34582 \pm 2.1862i$	$-0.025106 \pm 3.8928i$	$-0.010003 \pm 6.3345i$
$-2.4949 \pm 9.8315i$	$-0.27904 \pm 3.5116i$	$-0.025151 \pm 4.1968i$	$-0.010004 \pm 6.5290i$
$-2.5063 \pm 13.726i$	$-0.24678 \pm 4.5354i$	$-0.025212 \pm 4.4433i$	$-0.010004 \pm 6.6874i$
$-2.4971 \pm 17.155i$	$-0.25892 \pm 5.6822i$	$-0.025294 \pm 4.6597i$	$-0.010005 \pm 6.8269i$
$-2.5026 \pm 20.861i$	$-0.24631 \pm 6.7446i$	$-0.025408 \pm 4.8565i$	$-0.010005 \pm 6.9540i$
$-2.4984 \pm 24.300i$	$-0.25396 \pm 7.8720i$	$-0.025527 \pm 5.0397i$	$-0.010006 \pm 7.0720i$
$-2.5014 \pm 27.940i$	$-0.24762 \pm 8.9500i$	$-0.025762 \pm 5.2134i$	$-0.010006 \pm 7.1830i$
$-2.4990 \pm 31.389i$	$-0.25226 \pm 10.073i$	$-0.026085 \pm 5.3910i$	$-0.010007 \pm 7.2884i$
$-2.5009 \pm 34.997i$	$-0.24841 \pm 11.160i$	$-0.026198 \pm 5.5778i$	$-0.010008 \pm 7.3891i$

The first pair of eigenvalues corresponds to those of Gorodtsov and Leonov. They approach the continuous spectrum (i.e. the line segment from $-\frac{1}{W} - i\alpha$ to $-\frac{1}{W} + i\alpha$) as $W \rightarrow 0$. For large W , the real parts approach $-\frac{1}{2W}$ and the imaginary parts tend to $\pm\alpha$. For $R = 0$, this behavior can be deduced from the formula given in [3].

The remaining eigenvalues are basically lined up along the line $\text{Re } \sigma = -\frac{1}{2W}$. We see in the first two columns that the real parts approach $-\frac{1}{2W}$ as the imaginary parts increase, this is in fact suggested by the analysis for $\alpha = 0$. The eigenvalues in the third column appear to move away from $-\frac{1}{2W}$, but this trend is reversed as the imaginary parts increase further. For $W = 50$, a large number of Chebyshev modes (we used 120) was required to obtain sufficiently good results. All the eigenvalues at $W = 20$ and in particular at $W = 50$ have real parts close to $-\frac{1}{2W}$. This suggests that the real parts of the eigenvalues

approach $-\frac{1}{2W}$ at large W , as is indeed the case for $\alpha = 0$. The results tabulated above for $\alpha = 15$ suggest that the real parts of the eigenvalues also tend to $-\frac{1}{2W}$ at large α . The spacing between the imaginary parts of the eigenvalues decreases with increasing W , while the imaginary part of the second eigenvalue first decreases and then increases again. This behaviour would be expected from an analysis of characteristic wave speeds [14].

While the actual spectrum of the differential equation shows no instabilities, the numerical approximation does. In the numerical calculations at low Reynolds numbers, we can distinguish four clearly separated sets of eigenvalues:

1. Spurious eigenvalues: By this we mean eigenvalues of the discretized problem which do not approximate those of the differential equation even qualitatively and lie in a totally different part of the complex plane.
2. The two "Gorodtsov-Leonov" eigenvalues.
3. Eigenvalues approximating the remainder of the discrete spectrum.
4. Eigenvalues approximating the continuous spectrum, i.e. the line segment from $-\frac{1}{W} - i\alpha$ to $-\frac{1}{W} + i\alpha$.

In all our calculations we found four spurious modes. These spurious modes exist even in the Newtonian case [15] and they can be unstable. A certain number of the eigenvalues in the third group gives good approximations to those of the differential equation, but as the imaginary parts increase, the accuracy ultimately deteriorates. Since the real parts are small compared to the imaginary parts, they are particularly affected and they can in fact deteriorate to the point where they have the wrong sign, thus creating artificial instabilities. Not surprisingly, this is most likely to happen at high Weissenberg numbers. The approximation to the continuous spectrum is generally poor. This is not surprising. The method used here approximates isolated eigenvalues with C^∞ -eigenfunctions with infinite order accuracy [15], but this is not the case for a continuous spectrum. The approximation to the continuous spectrum is best near the ends and worst near the middle. Again artificial instabilities develop at high Weissenberg numbers. We believe that the "spurious" modes of Ho and Denn [1] also result from poor approximation to the continuous spectrum. The following table shows the number of numerically unstable eigenvalues as a function of the number of Chebyshev modes at $R = 1$, $\alpha = 1$ and $W = 20$.

No. of modes	30	40	50	60	80
Group 1	2	2	2	2	2
Group 3	8	10	14	16	22
Group 4	17	19	19	13	0

We see that the instabilities from the continuous spectrum disappear if a sufficient number of Chebyshev modes is used, while those from the discrete spectrum only get shifted towards higher imaginary parts, and the number of unstable eigenvalues actually increases.

The equations for *steady* flow of an upper convected Maxwell fluid undergo a change of type when $RW = 1 + W^2$ [14]. All the results reported above are subcritical, i.e. $RW < 1 + W^2$. As an example of a subcritical case, we looked at $R = 10$, $W = 2$, $\alpha = 5$. The eigenvalues with the lowest imaginary parts are as follows:

$-0.27302 \pm 0.1768i$
 $-0.28615 \pm 0.8463i$
 $-0.25010 \pm 0.9283i$
 $-0.24455 \pm 1.5281i$
 $-0.25267 \pm 2.0821i$
 $-0.25051 \pm 2.5991i$
 $-0.24896 \pm 3.0702i$
 $-0.25073 \pm 3.5151i$
 $-0.24993 \pm 3.9371i$
 $-0.24991 \pm 4.3364i$
 $-0.25015 \pm 4.7198i$
 $-0.35520 \pm 4.7784i$
 $-0.24969 \pm 5.0858i$
 $-0.25060 \pm 5.4386i$

The change of type means that in a part of the domain the speed of the fluid exceeds a characteristic wave speed. This manifests itself in the imaginary parts of the eigenvalues, which now "fill out" the whole real axis including the interval $(-\alpha, \alpha)$, rather than being separated by a gap in the middle. The qualitative behavior of the real parts, however, is unchanged; they are still close to $-\frac{1}{2W}$, except for the twelfth pair, which represents the Gorodtsov-Leonov eigenvalues.

We did a few calculations at a high Reynolds number ($R = 10000$, $\alpha = 1$). In the Newtonian case, the following asymptotic formula for the least stable eigenvalues holds for $R \rightarrow \infty$ (see [16], section 31.1):

$$\sigma = -1.0626\alpha^{2/3}R^{-1/3} \pm i(\alpha - 4.1288\alpha^{2/3}R^{-1/3}). \quad (22)$$

At $R = 10000$, $\alpha = 1$, this is equal to $-0.04932 \pm 0.80836i$. The following table shows our computed results at various values of W .

$W = 0.01$	$W = 0.1$	$W = 0.5$	$W = 2$
$-0.05205 \pm 0.81213i$	$-0.05167 \pm 0.81164i$	$-0.04984 \pm 0.80960i$	$-0.04157 \pm 0.80409i$

We see that elasticity has a destabilizing effect, but it does not seem to lead to instability. Apart from the spurious modes, we found no artificial instabilities of the numerical discretization, unless very few Chebyshev modes are used. Even though the numerical method is slower to converge at higher Reynolds numbers, it is less likely to produce unstable eigenvalues.

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